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Periodic and chaotic programs of optimal intertemporal allocation in an aggregative model with wealth effects*

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Summary. We examine a discrete-time aggregative model of discounted dynamic optimization where the felicity function depends on both consumption and capital stock. The need for studying such models has been stressed in the theory of optimal growth and also in the economics of natural resources. We identify conditions under which the optimal program is monotone. In our framework, the optimal program can exhibit cyclic behavior for all discount factors close to one. We also present an example to show that our model can exhibit optimal behavior which is chaotic in both topological and ergodic senses.

1. Introduction

In this paper, we examine a discrete-time aggregative model of discounted dynamic optimization where the felicity function depends on both consumption and capital stock. The need for studying such a model has been stressed in the theory of optimal growth and also in the economics of natural resources.

In his report on a number of studies in optimal economic growth, Koopmans (1967, p. 2) observed: "In all of the models considered it is assumed that the objective of economic growth depends exclusively on the path of *consumption* as foreseen for the future. That is, the capital stock is not regarded as an end in itself, or as a means to end other than consumption. We have already taken a step away from reality by making this assumption. A large and flexible capital stock has considerable importance for what is usually somewhat inadequately called "defense". The capital stock also helps to meet the cost of retaining all aspects of national sovereignty and power in a highly interdependent world." Interestingly enough, a continuous-time aggregative model in which the objective function is sensitive to both the consumption stream and to the per capita capital stock ("wealth effect") of the society

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was analyzed in a paper by Kurz (1968). It was noted there that the presence of wealth effects may lead to the appearance of multiple steady states. In fact, two numerical examples definitely established the possibility of such a phenomenon arising in dynamic optimization models. Some further work on a similar model was reported subsequently by Arrow and Kurz (1970).

In environmental economics, incorporating the direct welfare effects of certain types of capital assets [generally referred to as a "stock effect"] is acknowledged to be a significant ingredient in formulating appropriate resource policies. Dasgupta (1982, p. 107) summarizes the importance of such effects as follows: "As a flow DDT is useful in agriculture as an input; as a stock it is hazardous for health. Likewise fisheries and aquifers are useful not only for the harvest they provide: as a stock they are directly useful, since harvesting and extraction costs are low if stocks are large. Likewise, forests are beneficial not only for the flow of timber they can supply: as a stock they prevent soil erosion and maintain a genetic pool". To elaborate somewhat on the standard model of a commercially exploited fishery (which can be viewed as a special case of the aggregative model we study), the harvest is determined by a production function depending on the inputs of fishing effort and the stock of fish being exploited ("fish biomass"). This leads to a cost function (where cost is measured in terms of fishing effort) which depends on the harvest itself as well as the fish stock. If benefit from harvest is obtained through a specified revenue function, the net benefit or profit function is seen to be dependent on both the harvest and the stock of fish. [Clark (1976, Chapter 7) and Dasgupta (1982, Chapters 6, 7) develop this model in detail and also examine the nature of optimal harvest policies for some specific cases].

The aggregative model of optimal intertemporal allocation, in which felicity is derived solely from consumption (which is a special case of the model in this paper), has of course been studied extensively in the literature under a variety of different technological specifications. It is observed there that optimal programs exhibit monotone behavior over time. In fact, this feature continues to hold even when the technology exhibits increasing returns [see Dechert-Nishimura (1983), Mitra-Ray (1984)] and investment is irreversible [see Majumdar-Nermuth (1983)]. Thus, it is often suggested that a one-sector optimal growth model can only display "simple dynamics". The present exercise indicates that optimal programs in the aggregative model, in the presence of wealth (stock) effects, can exhibit the full spectrum of "complex dynamics" studied in the mathematical theory of chaos.

A few remarks on the relation of this result to those presented in the literature might help to put our paper in proper perspective. One of the major developments in the last ten years or so, in the area of dynamic economics, has been the identification of relatively simple non-linear models in which "cyclical" or "chaotic" outcomes are consistent with optimizing behavior over time based on complete information. [A number of excellent reviews of this development, from different perspectives, are available in Baumol and Benhabib (1989), Day and Pianigiani (1991), Grandmont (1985, 1986), Brock and Dechert (1991) and Boldrin and Woodford (1990)]. Much of this work has been carried out in the so-called "reduced form" model, in the tradition of Gale (1967) and McKenzie (1968). In this version, the principal object of interest is the immediate utility generated by a transition from a stock of goods in one period to another stock in the next period. That is, the focus of attention is the associated concept of the "reduced-form" utility function, which subsumes both the technology set and the welfare function, the "primitives" of the model. [For elaboration, see Section 3, especially 3b].

In the context of this (reduced form) model, Benhabib and Nishimura (1985) identified sufficient conditions for the existence of periodic optimal programs. However, commenting on the possibility of chaos in their model, they observed that "it is very difficult to construct examples which generate chaotic dynamics for infinite-horizon models that have concave utility functions." In subsequent papers, Deneckere and Pelikan (1986) and Boldrin and Montruchhio (1986b) provided the kind of examples that Benhabib-Nishimura were referring to. In particular, given any specified C^2 function Boldrin and Montruchhio (1986b) provided a method of constructing a reduced form model, such that the optimal policy function of the model would coincide with the pre-specified C^2 function. [Chaos is exhibited by choosing the C^2 function to be the logistic map, for instance].

The question naturally arises whether one can specify primitives (technology sets and welfare functions) satisfying standard assumptions, which will give rise to the reduced-form models constructed in the above exercises. It is clear from the above papers that one can specify dynamic optimization models (in terms of primitives) with *two production sectors* which generate the corresponding reduced-form models exhibiting cyclical [see Benhabib-Nishimura (1985)] and chaotic [see Deneckere and Pelikan (1986), Boldrin and Montrucchio (1986b)] optimal trajectories.

The approach taken in this paper is somewhat different from that of the above literature. We start with a model of dynamic optimization, specified in terms of primitives: the technology set, the welfare function and the discount factor. The welfare function is defined on consumption and the capital input stock, because the need for capturing such a stock or wealth effect has been repeatedly emphasized in the literature (as noted above) in a number of different intertemporal optimization problems. Our model is kept simple in every other respect, with the standard technological specification of the aggregative model. We are then interested in answering the following question: what kinds of dynamic optimal behavior can this model exhibit? In answering the question, we make considerable use of the results developed for the reduced model; however, our emphasis on the primitives of the particular dynamic optimization model we study is the dominant theme throughout.

Our answer to the above question can be conveniently subdivided into three parts. First, we indicate briefly [by relying on the analysis of Benhabib-Nishimura (1985)] how sufficient conditions on the welfare and production functions can be developed to yield a monotone non-decreasing optimal policy function [see Section 4a]. We also examine in some detail [in Section 4b] the restrictiveness of assumptions which have typically been made directly on the reduced-form model to ensure a monotone non-increasing optimal policy function (in the interior of the transition possibility set). The implications of such assumptions on the underlying welfare and production functions need not be at all transparent, as we proceed to show. The methodological point that emerges from this exercise is the need to formalize conditions on the reduced model in terms of justifiable restrictions on the primitive data [see Section 4 for details].

Second, we show [in Section 5] that our model can exhibit optimal cyclic behavior for *all* discount factors close to one. Clearly, this goes against the spirit of turnpike theorems which propose to show that for discount factors sufficiently close to one, one has global asymptotic stability of optimal programs with respect to the stationary optimal stock. It is, of course, perfectly in harmony with the more general result of the subject, namely the "neighborhood turnpike theorem" of McKenzie (1982). As the discount factor approaches one, the cycles of optimal programs are seen to exhibit smaller amplitude around the stationary optimal stock.

Third, we construct a suitable example [in Section 6a] to demonstrate that our model can exhibit optimal behavior, which is chaotic in the topological and ergodic senses. Unlike the literature preceding this paper, we go a step further [in Section 6b] and enquire whether the (demonstrated) chaotic behavior is robust to small perturbations of the "economy", specified in terms of the primitives (the production and welfare functions and the discount factor). We show that topological chaos is robust to such perturbations by demonstrating that the optimal policy function of each "neighboring" economy satisfies the Li-Yorke condition.

2. Basic results from the theory of dynamical systems

A *dynamical system* is described by a pair (X, h), where X is a set (called the *state space*), and h a function from X to X describing the *law of motion* of the state variable, $x \in X$. Thus, if $x_t \in X$ is the state of the system in period t, then $x_{t+1} = h(x_t) \in X$ is the state of the system in period $(t + 1)$. We will be concerned, in what follows, with dynamical systems for which X is an interval of the real line.

Given a dynamical system (X, h), and any $x \in X$, we define $h^0(x) = x$, and for any integer $k \geq 1$

$$h^k(x) = h(h^{k-1}(x))$$

A point $x \in X$ is called *periodic* if there is $k \geq 2$ such that $h^k(x) = x$. The smallest such k, call it $\mathbf{k}$, is the *period* of x. [In this case, the sequence $\{h^j(x)\}_0^\infty$ is also called periodic with period $\mathbf{k}$]. We denote the set of periodic points in X by $P(X)$. Its complement in X, the set of non-periodic points, is denoted by $N(X)$.

A basic result characterizing the behavior of the dynamical system (X, h) has been given by Li and Yorke (1975), and may be stated as follows.

Theorem 2.1

Let α, β be in $\mathscr{R}$, with $\alpha < \beta$. Suppose $X = [\alpha, \beta]$ and $h: X \to X$ is continuous. If there is $x^* \in X$ such that

$$h^3(x^*) \leq x^* < h(x^*) < h^2(x^*) \qquad \text{(L-Y)}$$

then

(i) for every integer $\mathbf{k} \geq 2$, there is a periodic point $x_k \in X$ with period $\mathbf{k}$;
(ii) there is an uncountable set $W \subset N(X)$ satisfying the following conditions:

(a) *If* $x, y \in W$ *with* $x \neq y$, *then* $\overline{\lim}_{k\to\infty} |h^k(x) - h^k(y)| > 0$; $\underline{\lim}_{k\to\infty} |h^k(x) - h^k(y)| = 0$.

(b) *If* $x \in W$ *and* $y \in P(X)$, *then* $\overline{\lim}_{k\to\infty} |h^k(x) - h^k(y)| > 0$.

REMARKS:

(1) The result (i) is actually a consequence of Sarkovskii's theorem (see Devaney [1989]).

(2) According to (ii)(a), there are pairs of initial states (in W) such that the sequences of iterates move apart and return close to each other infinitely often. Furthermore, as (ii)(b) states, if the initial state is in the set W, then the system does not converge to any periodic point.

We will say that the dynamical system (X, h) exhibits *topological chaos* if conditions (i) and (ii) of Theorem 2.1 are satisfied. The Li-Yorke condition (L-Y) can be seen as an easily verifiable sufficient condition for topological chaos.

It has been argued that topological chaos may be "unobservable" since the uncountable set W in (ii) of Theorem 2.1 may have Lebesgue measure zero [see, for example, Collet and Eckmann (1980), Day and Shafer (1987) for discussions]. This motivates the study of "ergodic chaos" which we turn to next.

Let μ denote the Lebesgue measure on X (endowed with its Borel σ-algebra). We will say that the dynamical system (X, h) exhibits *ergodic chaos* if there exists a probability measure ν satisfying the following conditions:

(i) ν is *absolutely continuous* with respect to the Lebesgue measure on X; that is, if B is a Borel set in X, and $\mu(B) = 0$ then $\nu(B) = 0$;

(ii) ν is *invariant* under the action of h; that is, $\nu(h^{-1}(B)) = \nu(B)$ for every Borel set $B \subset X$;

(iii) ν is *ergodic*; that is, for every ν-integrable real valued function, ϕ (on X), we have

$$\lim_{T\to\infty} (1/T) \sum_{k=1}^{T} \phi(h^k(x)) \to \int \phi \, d\nu \quad \text{for } \nu - \text{a.e. } x \in X.$$

Condition (iii) above is frequently referred to as: time averages equal space averages. If ν is the unique probability measure which satisfies (i)–(iii) above, it is called *the ergodic measure* of h.

If (X, h) has an ergodic measure ν, then for ν – a.e. $x \in X$ (and hence, by (i), for x belonging to a subset of X with positive Lebesgue measure) the sequence of iterates $\{h^k(x)\}$ will "fill up" the support of the measure ν, and as a consequence, will have extremely complicated trajectories.

In order to provide sufficient conditions for ergodic chaos, we introduce the notion of a Schwarzian derivative [see Devaney (1989) for a more extensive discussion]. Consider a dynamical system (X, h) with $h: X \to X$ of class C^3. The *Schwarzian derivative* $Sh(x)$ is given by

$$Sh(x) = [h'''(x)/h'(x)] - (3/2)[h''(x)/h'(x)]^2$$

for $x \in X$ with $h'(x) \neq 0$.

Theorem 2.2:

Let α, β be in $\mathscr{R}$ with $\alpha < \beta$. Suppose $X = [\alpha, \beta]$ and $h: X \to X$ satisfy the following conditions:

(i) h is of class C^3, and there exists $x^* \in (\alpha, \beta)$ such that $h'(x^*) = 0$ and $h''(x^*) < 0$; $h'(x) > 0$ for all $x \in X$ with $x < x^*$; $h'(x) < 0$ for all $x \in X$ with $x > x^*$.

(ii) $h(x) > x$ for all $x \in (\alpha, x^*)$; $h(x^*) \in (x^*, \beta]$; and $Sh(x) < 0$ for all $x \in X$ except $x = x^*$.

(iii) There exists $k \geq 2$ such that $y = h^k(x^*)$ satisfies $h(y) = y$ and $|h'(y)| > 1$.

Then (X, h) exhibits ergodic chaos.

A detailed discussion of Theorem 2.2 may be found in Grandmont (1986) and Day and Pianigiani (1991).

Recall from the definition of topological chaos for a dynamical system (X, h), that condition (ii) indicates a certain "sensitive" dependence on initial conditions. Roughly speaking, the extent of this sensitive dependence is measured by the Lyapunov exponent, which we turn to next.

Let (X, h) be a dynamical system, with $h: X \to X$ of class C^1. For any $x_0 \in X$, the *Lyapunov exponent*, $\lambda(x_0)$, is defined as

$$\lambda(x_0) = \lim_{k \to \infty} (1/k) \, ln \left| \frac{dh^k}{dx}(x_0) \right|$$

For sufficiently large k and small $\varepsilon > 0$, the Lyapunov exponent "approximately satisfies"

$$\varepsilon e^{k\lambda(x_0)} \approx |h^k(x_0 + \varepsilon) - h(x_0)|$$

The right-hand side indicates how far apart x_0 and $x_0 + \varepsilon$ are under k iterates of h. When $\lambda(x_0) > 0$, initially nearby points are stretched (by the successive iterations of h) at a positive exponential rate. [Further discussion of the Lyapunov exponent can be found in Rasband (1990)].

One of the most celebrated examples of a dynamical system (X, h) is one where $X = [0, 1]$ and

$$h(x) = 4x(1 - x) \quad \text{for all } x \in X$$

It is not difficult to verify that for this example, Theorems 2.1 and 2.2 apply directly. [For Theorem 2.1, choose $x^* = [\sqrt{2} - 1]/2\sqrt{2}$ to verify the Li-Yorke condition (L-Y). For Theorem 2.2, choose $x^* = (1/2)$, and note that the Schwarzian derivative $Sh(x) = -6/(1 - 2x)^2$ for all $x \neq (1/2)$, to verify (i) and (ii); further (iii) is verified by choosing $k = 2$]. Thus, this dynamical system exhibits both topological and ergodic chaos.

The density $Q(x)$ of the ergodic measure of this dynamical system has been computed. It is given by

$$Q(x) = [\pi \sqrt{x(1 - x)}]^{-1} \quad \text{for } 0 < x < 1$$

[See Day and Pianigiani (1991) for more on this computation, due to Ulam and von Neumann]. The Lyapunov exponent of this dynamical system is in fact seen to be independent of $x_0 \in X$, and is given by

$$\lambda = \lambda(x_0) = ln 2 > 0 \quad \text{for all } x_0 \in X$$

3. Preliminary aspects of the optimization framework

3a. The model

We consider an aggregative model, specified by a *production function*, $f: \mathscr{R}_+ \to \mathscr{R}_+$, a *welfare function*, $w: \mathscr{R}_+^2 \to \mathscr{R}_+$, and a *discount factor* $\delta \in (0, 1)$.

The following assumptions on f are used:

(F.1) $f(0) = 0$; *f is continuous* on $\mathscr{R}_+$.
(F.2) f is non-decreasing and concave on $\mathscr{R}_+$.
(F.3) There is some $K > 0$ such that $f(x) > x$ when $0 < x < K$, and $f(x) < x$ when $x > K$.

We define a set $\Omega \subset \mathscr{R}_+^2$ as follows:

$$\Omega = \{(x, z) \varepsilon \mathscr{R}_+^2 : z \leq f(x)\}$$

The following assumptions on w are used:

(W.1) $w(x, c)$ is continuous on $\mathscr{R}_+^2$.
(W.2) $w(x, c)$ is non-decreasing in x given c, and non-decreasing in c, given x on $\mathscr{R}_+^2$. Furthermore, if $x > 0$, $w(x, c)$ is strictly increasing in c on Ω.
(W.3) $w(x, c)$ is concave on $\mathscr{R}_+^2$. Furthermore, if $x > 0$, $w(x, c)$ is strictly concave in c on Ω.

A *program* from $\mathbf{x} \geq 0$ is a sequence $\{x_t\}_0^\infty$ satisfying

$$x_0 = \mathbf{x}, 0 \leq x_{t+1} \leq f(x_t) \quad \text{for } t \geq 0$$

The *consumption* sequence $\{c_{t+1}\}_0^\infty$ is given by

$$c_{t+1} = f(x_t) - x_{t+1} \quad \text{for } t \geq 0$$

It is easy to verify that for any program $\{x_t\}_0^\infty$ from $\mathbf{x} \geq 0$, we have $x_t, c_{t+1} \leq K(\mathbf{x}) \equiv \max(K, \mathbf{x})$ for $t \geq 0$. [In particular, if $\mathbf{x} \in [0, K]$, then $x_t, c_{t+1} \leq K$ for $t \geq 0$.]

A program $\{\hat{x}_t\}_0^\infty$ from $\mathbf{x} \geq 0$ is *optimal* if

$$\sum_{t=0}^{\infty} \delta^t w(\hat{x}_t, \hat{c}_{t+1}) \geq \sum_{t=0}^{\infty} \delta^t w(x_t, c_{t+1})$$

for every program $\{x_t\}_0^\infty$ from $\mathbf{x}$.

A program $\{x_t\}_0^\infty$ from $\mathbf{x}$ is *stationary* if $x_t = \mathbf{x}$ for $t \geq 0$. It is a *stationary optimal program* if it is also an optimal program from $\mathbf{x}$. In this case, $\mathbf{x}$ is called a *stationary optimal stock.* [Note that 0 is a stationary optimal stock]. A stationary optimal stock, x, is *non-trivial* if $x > 0$.

3b. Conversion to reduced form

The model, described in the previous subsection, (which we will call the "primitive form") can be converted to the so-called "reduced form". In reduced form, an intertemporal framework is described by a state space, a transition possibility set, a (reduced form) utility function defined on this set, and a discount factor.

The state space for our purpose will be taken to be $\mathscr{R}_+$, and we will think of the input (x) as the state variable. The transition possibility set then describes pairs of input levels (x, z), such that if x is the input level today (at t), then it is possible to go to the input level z tomorrow (at $(t+1)$). The utility function then measures the welfare obtained in moving from x today to z tomorrow.

Now, we describe how our framework in primitive form can be viewed as a special case of the reduced form model. Note that Ω, as defined in Section 2a, precisely describes the transition possibility set. The utility function $u:\Omega \to \mathscr{R}_+$ can be defined by

$$u(x, z) = w(x, f(x) - z)$$

Under the assumptions (F.1)–(F.3) on the production function f, the transition possibility set Ω satisfies the following properties:

(Ω.1) $(0,0) \in \Omega$; $(0, z) \in \Omega$ implies $z = 0$.
(Ω.2) Ω is a closed, convex subset of $\mathscr{R}^2_+$.
(Ω.3) If $(x, z) \in \Omega$ and $x' \geq x$ and $0 \leq z' \leq z$, then $(x', z') \in \Omega$.
(Ω.4) There is some $K > 0$, such that if $(x, z) \in \Omega$ and $x > K$, then $z < x$.

Furthermore, under the assumptions (W.1)–(W.3) on the welfare function w, the (reduced form) utility function u satisfies the following properties

(U.1) $u(x, z)$ is continuous on Ω.
(U.2) If $(x, z) \in \Omega$ and $x' \geq x$ and $0 \leq z' \leq z$, then $u(x', z') \geq u(x, z)$.
(U.3) $u(x, z)$ is concave on Ω; and, given $x > 0$, $u(x, z)$ is strictly concave in z on Ω.

The concepts of a program and that of an optimal program (introduced in Section 3a) can then be viewed equivalently as follows. A program $\{x_t\}_0^\infty$ from $\mathbf{x} \geq 0$ is seen to be defined by

$$x_0 = \mathbf{x}, (x_t, x_{t+1}) \in \Omega \quad \text{for } t \geq 0$$

And an optimal program $\{\hat{x}_t\}_0^\infty$ from $\mathbf{x} \geq 0$ is seen to be a program satisfying

$$\sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}) \geq \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

for every program $\{x_t\}_0^\infty$ from $\mathbf{x}$.

This conversion of our framework in primitive form to the reduced form is particularly helpful because it allows a direct application of the many results developed for the reduced form model [see McKenzie (1986) for a comprehensive survey of the more important results]. We would like to emphasize, however, that our assumptions are exclusively on the *primitive form*, and these *imply* that certain properties are satisfied by the corresponding *reduced form*. If some property is assumed about (Ω, u, δ), this needs to be justified by appropriate assumptions on the primitives (f, w, δ) of the model. We will discuss this point in more detail in Section 4.

3c. Value and policy functions

Given a reduced form model (Ω, u, δ), satisfying (Ω.1)–(Ω.4), (U.1)–(U.3), standard arguments ensure that there is an optimal program from every $x \geq 0$. Conditions (Ω.2) and (U.3) ensure that an optimal program is unique.

We define a *value function* $V: \mathscr{R}_+ \to \mathscr{R}$ by

$$V(x) = \sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}) \tag{3.1}$$

and the *optimal policy function* $h: \mathscr{R}_+ \to \mathscr{R}_+$ by

$$h(x) = \hat{x}_1 \tag{3.2}$$

where $\{\hat{x}_t\}$ is the optimal program from $x \geq 0$.

It is easy to check that V is a concave function on $\mathscr{R}_+$. Furthermore, an application of an appropriate version of the maximum theorem yields the continuity of V and h on $\mathscr{R}_+$ [see, for example, Dutta-Mitra (1986)].

Given any $x \in \mathscr{R}_+$, the constrained maximization problem

$$\begin{array}{ll} \text{Max} & [u(x,z) + \delta V(z)] \\ \text{Subject to} & (x,z) \in \Omega \end{array}$$

has a unique solution, $H(x)$. It follows from the definitions of V and h that $H(x) = h(x)$ for all $x \in \mathscr{R}_+$ and further $V(x) = u(x, h(x)) + \delta V(h(x))$. This yields the "optimality equation" for each $x \in \mathscr{R}_+$:

$$V(x) = \max_{(x,z) \in \Omega} [u(x,z) + \delta V(z)]$$

It is not difficult to verify that if ψ is any continuous real valued function on $[0, K]$ satisfying the "functional equation of dynamic programming":

$$\psi(x) = \max_{(x,z) \in \Omega} [u(x,z) + \delta \psi(z)] \quad \text{for } x \geq 0$$

then $\psi(x) = V(x)$ for all $x \in [0, K]$. That is, V is the *unique* solution (in the class $C[0, K]$) of the functional equation of dynamic programming.

We summarize the above facts in the following theorem, for ready reference.

Theorem 3.1:

(i) The value function, V, defined in (3.1) is the unique continuous real valued function on $[0, K]$, satisfying the functional equation of dynamic programming:

$$V(x) = \max_{(x,z) \in \Omega} [u(x,z) + \delta V(z)] \tag{3.3}$$

Further, V is concave and non-decreasing on $\mathscr{R}_+$.

(ii) The policy function h, defined in (3.2) satisfies the following property: for each $x \in \mathscr{R}_+$, $h(x)$ solves uniquely the constrained maximization problem

$$\begin{array}{ll} \text{Max} & [u(x,z) + \delta V(z)] \\ \textit{Subject to} & (x,z) \in \Omega \end{array}$$

Further, h is continuous on $\mathscr{R}_+$.

3d. Dual variables associated with optimal programs

In the course of this paper, we will have to show at various points that certain specified programs are optimal. One way of doing this is by finding a value function satisfying the functional equation of dynamic programming. (We use this method to verify a result of Section 6). Another way is by specifying a sequence of dual variables (or "shadow prices") associated with the program, such that the program is "competitive" and satisfies the "transversality condition". (We use this method to verify the result of Section 5). We state this well-known result for ready reference.

Theorem 3.2:
Suppose $\{\hat{x}_t\}_0^\infty$ is a program from **x**, *and $\{\hat{p}_t\}_0^\infty$ is a sequence of non-negative numbers such that*

(i) $\delta^t u(\hat{x}_t, \hat{x}_{t+1}) + \hat{p}_{t+1}\hat{x}_{t+1} - \hat{p}_t\hat{x}_t \geq \delta^t u(x, z) + \hat{p}_{t+1}z - \hat{p}_t x$
for all $(x, z) \in \Omega$ and all $t \geq 0$

and

(ii) $\lim_{t \to \infty} \hat{p}_t \hat{x}_t = 0.$

Then $\{\hat{x}_t\}_0^\infty$ is an optimal program from **x**.

4. Observations on the primitive and reduced forms

A considerable part of the literature on optimal intertemporal allocation has been developed for the reduced-form model. In this section, we emphasize the need to look at the primitive model, which gives rise to the reduced form, in order to assess the importance and compatability of some assumptions which are typically made directly on the reduced form.

For this purpose, we consider a reduced form model (Ω, u, δ) as described in Section 3b, with Ω satisfying (Ω.1)–(Ω.4), u satisfying (U.1)–(U.3), and $0 < \delta < 1$ and proceed to examine situations for which the optimal policy function will be monotone.

4a. Monotone increasing policy functions

We first examine the sufficient conditions which ensure that the optimal policy function is monotone non-decreasing. This case has traditionally been of importance because it leads to monotone behavior of optimal input stocks, converging to some stationary optimal input stock.

Denote the interior of Ω by Ω^0. We say that u is *supermodular on Ω^0* if whenever (x', z'), (x'', z''), (x', z'') and (x'', z') belong to Ω^0 and $(x'', z'') \gg (x', z')$, we have

$$u(x'', z'') + u(x', z') \geq u(x'', z') + u(x', z'') \qquad (\bar{S})$$

Following Ross (1983) and Benhabib-Nishimura (1985), it can be shown that if u is supermodular on Ω^0, then h is non-decreasing on $\mathscr{R}_+$. It is worthwhile to state this result formally.

Proposition 4.1
Suppose (Ω, u, δ) *satisfies* (Ω.1)–(Ω.4), (U.1)–(U.3) *and* $0 < \delta < 1$. *If u is supermodular on* Ω^0, *then the optimal policy function, h, is non-decreasing on* $\mathscr{R}_+$.

There is a convenient way to check the supermodularity of u on Ω^0. If u is C^2 on Ω^0 and $D_{12}u(x,z) \geq 0$ for all $(x,z) \in \Omega^0$, then [following Ross (1983)] u can be shown to be supermodular on Ω^0.

If we turn now to our primitive model, we can try to find conditions on (f, w, δ), which will ensure that u is C^2 on Ω^0 with $D_{12}u(x,z) \geq 0$ for all $(x,z) \in \Omega^0$. If f is C^1 on $\mathscr{R}_{++}$, and $w(x,c)$ is C^2 on Ω^0, one can verify that for any $(x,z) \in \Omega^0$,

$$D_{12}u(x,z) = [-D_{22}w(x, f(x)-z)]f'(x) - D_{12}w(x, f(x)-z)$$

Thus, if we assume that

$$-D_{22}w(x,c)f'(x) \geq D_{12}w(x,c) \quad \text{for all } (x,c) \in \Omega^0 \tag{4.1}$$

we can ensure $D_{12}u(x,z) \geq 0$ for all $(x,z) \in \Omega^0$. This leads to the following result.

Proposition 4.2
Suppose (f, w, δ) *satisfy* (F.1)–(F.3), (W.1)–(W.3), *and* $0 < \delta < 1$. *Assume, further, that f is* C^1 *on* $\mathscr{R}_{++}$, *w is* C^2 *on* Ω^0 *and (4.1) is satisfied. Then the optimal policy function, h, is non-decreasing on* $\mathscr{R}_+$.

That all the assumptions of Proposition 4.2 are compatible can be seen by considering the example where $f(x) = 2x^{1/2}$ for $x \in \mathscr{R}_+$, $w(x,c) = x^{1/2} + c^{1/2}$ for $(x,c) \in \mathscr{R}^2_+$ and $0 < \delta < 1$.

Under the assumptions of Proposition 4.2, if $\{\hat{x}_t\}_0^\infty$ is an optimal program from $\mathbf{x} > 0$, it is clearly a monotone (non-increasing or non-decreasing) sequence. Thus, the assumptions imply that optimal behavior leads to what can be called "simple dynamics".

4b. Monotone decreasing policy functions

It might appear that, symmetric to our analysis in Section 4a, we should be able to get sufficient conditions under which the optimal policy function is monotone non-increasing. However, this turns out to be incorrect except in the extreme case where h is an "extinction" policy (that is, $h(x) = 0$ for all $x \geq 0$). [A strong sufficient condition which ensures that h is *not* an extinction policy is the existence of a non-trivial stationary optimal stock.]

Start, again, with the reduced form model (Ω, u, δ) where Ω satisfies (Ω.1)–(Ω.4), u satisfies (U.1)–(U.3), and $0 < \delta < 1$.

Now, suppose h were monotone non-increasing for all $x > 0$ and there was some $\tilde{x} > 0$ for which $h(\tilde{x}) > 0$. Then, for all $0 < x < \tilde{x}$, $h(x) \geq h(\tilde{x})$ since h is monotone non-increasing. By continuity of h on $\mathscr{R}_+$, $h(0) \geq h(\tilde{x}) > 0$. But clearly this contradicts the impossibility of free production (Ω.1). It is worthwhile to state this formally.

Proposition 4.3
Suppose (Ω, u, δ) *satisfies* (Ω.1)–(Ω.4), (U.1)–(U.3), *and* $0 < \delta < 1$. *If h is the optimal policy function, and* $h(\tilde{x}) > 0$ *for some* $\tilde{x} > 0$, *h cannot be non-increasing for all* $x \in (0, \tilde{x}]$.

To relate this result to the sufficient conditions provided in the literature to ensure (locally) non-increasing policy functions, we can proceed as follows. Define u to be *submodular on* Ω^0 if whenever (x', z'), (x'', z''), (x', z'') and (x'', z') belong to Ω^0 and $(x'', z'') \gg (x', z')$, we have

$$u(x'', z'') + u(x', z') \leq u(x'', z') + u(x', z'') \tag{S}$$

Following Benhabib-Nishimura (1985), it can be shown that if u is submodular on Ω^0, and $(x^0, h(x^0)) \in \Omega^0$, then h is locally non-increasing at x^0.

Proposition 4.4

Suppose (Ω, u, δ) *satisfies* (Ω.1)–(Ω.4), (U.1)–(U.3), $0 < \delta < 1$ *and* u *is submodular on* Ω^0. *If for some* $\tilde{x}$, *we have* $(\tilde{x}, h(\tilde{x})) \in \Omega^0$, *then there is a neighborhood* $N(\tilde{x})$ *of* $\tilde{x}$, *such that the optimal policy function,* h, *is non-increasing on* $N(\tilde{x})$.

The submodularity of u on Ω^0 can be checked as follows. If u is C^2 on Ω^0 and $D_{12}u(x, z) \leq 0$ for all $(x, z) \in \Omega^0$, then u can be shown to be submodular on Ω^0.

Turning to our primitive model, we can try to find conditions on (f, w, δ) which will ensure that u is C^2 on Ω^0 with $D_{12}u(x, z) \leq 0$ for all $(x, z) \in \Omega^0$. If f is C^1 on $\mathscr{R}_{++}$, and $w(x, c)$ is C^2 on Ω^0, and for all $(x, c) \in \Omega^0$

$$-D_{22}w(x, c)f'(x) \leq D_{12}w(x, c) \tag{4.2}$$

we can ensure $D_{12}u(x, z) \leq 0$ for all $(x, z) \in \Omega^0$.

Now, we can make the following important observation. Condition (4.2) is inconsistent with the following condition:

$$\textit{There is some } \hat{x} > 0 \textit{ such that } (x, h(x)) \in \Omega^0 \textit{ for } x \in (0, \hat{x}] \tag{4.3}$$

For if (4.3) is satisfied, then by Proposition 4.4, we would have a non-increasing optimal policy function for all $x \in (0, \hat{x}]$. And that is impossible by Proposition 4.3. Thus, *any condition which ensures that (4.3) holds must contradict (4.2).*

A further implication of the fact that we have just noted may be obtained as follows. Suppose we assume condition (4.2), and there is some $\tilde{x} > 0$ such that $h(\tilde{x}) > 0$ [so that the optimal policy is not the "extinction" policy]. Then for some $0 < x \leq \tilde{x}$, we must have $h(x) = f(x)$; that is *for some positive input level, the optimal consumption level must be zero.* We note this formally in the next result.

Proposition 4.5

Suppose (f, w, δ) *satisfy* (F.1)–(F.3), (W.1)–(W.3), *and* $0 < \delta < 1$. *Assume, further that* f *is* C^1 *on* $\mathscr{R}_{++}$, w *is* C^2 *on* Ω^0, *and (4.2) is satisfied. If* $h(\tilde{x}) > 0$ *for some* $\tilde{x} > 0$, *then there is some input level* $x' > 0$, *for which* $h(x') = f(x')$.

If we assume the following Inada-type condition:

$$D_2w(x, \lambda c) \to \infty \text{ as } \lambda \to 0 \quad \textit{for all } (x, c) \in \Omega^0 \tag{4.4}$$

then we can ensure that $h(x) < f(x)$ for all $x > 0$. Thus, by Proposition 4.5, Condition (4.2) is inconsistent with Condition (4.4), whenever h is not the extinction policy.

Corollary 4.1

Under the assumptions of Proposition 4.5, if $h(\tilde{x}) > 0$ *for some* $\tilde{x} > 0$, *Condition (4.4) must be violated.*

It is worth emphasizing that the relationships between Conditions (4.2) and (4.3) and between Conditions (4.2) and (4.4), might not be readily transparent; they only become so after looking at the *implications* of these conditions for the nature of the optimal policy function.

We make a final remark before concluding this section. In the context of our model, the important contribution of Nishimura and Yano (1994) can be related to the preceding discussion as follows. If we suppose (in addition to the standard assumptions) that f is C^1 on $\mathscr{R}_{++}$, w is C^2 on Ω^0, (4.2) is satisfied, and there is a non-trivial stationary optimal stock, then something considerably stronger can be shown than is indicated in Proposition 4.5; viz., there is some input level $x' > 0$ such that $h(x) = f(x)$ for all $0 \le x \le x'$, and $h(x) < f(x)$ for all $x > x'$ [and, so, h is non-increasing for $x > x'$ by Proposition 4.4]. This means that the graph of h resembles a "tent-map", and so an interesting route to obtaining cyclical and chaotic optimal programs in this model is indicated, which is quite distinct from the technique pursued by Boldrin and Montruchhio (1986b).

5. Periodic optimal programs: an example

Beyond the "simple dynamics" discussed in Section 4a, what kinds of optimal dynamic behavior can our model exhibit? In this section, we show by constructing a suitable example, that our model can generate optimal cyclic behavior for *all* discount factors close to one. Examples with a similar flavor have been presented in *multi-commodity* models by Weitzman [reported in Samuelson (1973) and discussed by McKenzie (1983), Benhabib-Nishimura (1985) and others], and Wan (1988). These models can, in fact, be reduced to the form we have discussed in Section 3b. Indeed, the strategy used in obtaining our example is to start with the reduced-form version of Weitzman's example, and place assumptions on the primitives of our *aggregative* model which will yield that reduced form.

We proceed now to describe the example more precisely. Our purpose is to show that there is a specification of $(f, w, \hat{\delta})$ where

(i) $f: \mathscr{R}_+ \to \mathscr{R}_+$ satisfies (F.1)–(F.3);
(ii) $w: \mathscr{R}_+^2 \to \mathscr{R}_+$ satisfies (W.1)–(W.3);
(iii) $0 < \hat{\delta} < 1$

such that for *every* δ satisfying $\hat{\delta} < \delta < 1$,

(a) there is a unique non-trivial stationary optimal stock $x(\delta)$
(b) there is an open interval, $A(\delta)$, containing $x(\delta)$, such that if $\{x_t\}_0^\infty$ is the optimal program from $\mathbf{x} \in A(\delta)$, with $\mathbf{x} \neq x(\delta)$, then $\{x_t\}_0^\infty$ is periodic with period 2.

Thus, optimal programs continue to be periodic even for discount factors, δ, arbitrarily close to 1, and also for initial input stocks, $\mathbf{x}$, arbitrarily close to the stationary optimal stock, $x(\delta)$.

It is worth mentioning that this example might appear to contradict what is usually called a "turnpike theorem" for discount factors close to one. Actually, it does not, since a correct statement of such a turnpike theorem always includes an assumption of *strict concavity* of the reduced utility function on the transition possibility set. Our example (and Weitzman's) does not have a strictly concave reduced utility function. It should also be observed that the more general theorem of the subject, namely the "neighborhood turnpike theorem" [emphasized notably by McKenzie (1982, 1986)] is completely consistent with our example, since as the discount factor approaches one, optimal programs (even though they exhibit cyclic behavior) are confined in smaller and smaller neighborhoods of the corresponding stationary optimal stock.

Example 5.1:
Define $f:\mathscr{R}_+ \to \mathscr{R}_+$ by

$$f(x) = \begin{cases} (32/3)x - 32x^2 + (256/3)x^4 & \text{for } 0 \leq x < 0.25 \\ 1 & \text{for } x \geq 0.25 \end{cases}$$

Define $w:\mathscr{R}_+^2 \to \mathscr{R}_+$ by

$$w(x, c) = 2x^{1/2}c^{1/2} \quad \text{for all } (x, c) \in \mathscr{R}_+^2$$

Finally, define $\hat{\delta} = (1/3)$.

It can be checked that f satisfies (F.1)–(F.3) with $K = 1$, and w satisfies (W.1)–(W.3).

Now, let δ be any discount factor satisfying $\hat{\delta} < \delta < 1$. We fix this δ in what follows. Define an open interval

$$A(\delta) = \{x: 0.25 < x < 3\delta^2/(1 + 3\delta^2)\}$$

and also a point

$$x(\delta) = \delta/(1 + \delta)$$

It can be shown then that

(a) $x(\delta)$ is the unique non-trivial stationary optimal stock [note that $x(\delta) \in A(\delta)$];
(b) for every $\mathbf{x} \in A(\delta)$, with $\mathbf{x} \neq x(\delta)$, the optimal program $\{\hat{x}_t\}_0^\infty$ from $\mathbf{x}$ is periodic with period 2.

6. Chaotic optimal programs

6a. An example of an economy exhibiting topological and ergodic chaos

Optimal cyclic behavior, though "less simple" than monotone behavior, has considerable regularity to it to be viewed as "predictable". In this section, we show by constructing a suitable example that our model can exhibit optimal behavior which is chaotic (in both the "topological" and "ergodic" senses).

Examples similar to ours have been constructed, among others, by Deneckere and Pelikan (1986) and Boldrin and Montruchhio (1986b) in the context of the reduced-form model discussed in Section 3b. These models in turn have been shown to be reduced-form versions of *two-sector models*. In our example, an *aggregative model* is seen to yield the reduced-form for which the optimal policy function is the logistic function (for an appropriate domain of inputs).

It is, perhaps, worth mentioning that an example of chaotic optimal behavior is obtained in an aggregative model (with wealth effect) by Boldrin and Montrucchio (1986a). While the framework is superficially similar to ours, the production function in their example, is, in fact, "bell-shaped" (initially strictly increasing and thereafter strictly decreasing) and this feature appears to be unattractive in terms of economic modeling, as well as crucial to their construction.

We now describe the nature of our example more formally. Our purpose is to show that there is a specification of $(f, \mathbf{w}, \bar{\delta})$ where

(i) $f:\mathscr{R}_+ \to \mathscr{R}_+$ satisfies (F.1)–(F.3);
(ii) $\mathbf{w}:\mathscr{R}^2_+ \to \mathscr{R}_+$ satisfies (W.1)–(W.3);
(iii) $0 < \bar{\delta} < 1$

such that the optimal policy function, $\mathbf{h}:\mathscr{R}_+ \to \mathscr{R}_+$ satisfies the logistic equation

$$\mathbf{h}(x) = 4x(1-x) \quad \text{for } 0 \le x \le 1$$

Example 6.1:
Define $f:\mathscr{R}_+ \to \mathscr{R}_+$ by:

$$f(x) = \begin{cases} (16/3)x - 8x^2 + (16/3)x^4 & \text{for } x \in [0, 0.5) \\ 1 & \text{for } x \ge 0.5 \end{cases}$$

$$\bar{\delta} = 0.0025$$

The function $\mathbf{w}:\mathscr{R}^2_+ \to \mathscr{R}_+$ is defined in a more involved fashion. Define the parameters $L = 98$, $a = 425$. Denote by I the closed interval $[0, 1]$, and define the function $\theta:I \to I$ by

$$\theta(x) = 4x(1-x) \quad \text{for } x \in I$$

Now, define $\mathbf{u}:I^2 \to \mathscr{R}$ by

$$\mathbf{u}(x, z) = ax - 0.5Lx^2 + z\theta(x) - 0.5z^2 - \bar{\delta}[az - 0.5Lz^2 + 0.5\theta(z)^2]$$

Define the set $\mathbf{D} \subset I^2$ by

$$\mathbf{D} = \{(x, c) \in I \times \mathscr{R}_+ : c \le f(x)\}$$

and a function $\mathbf{w}:\mathbf{D} \to \mathscr{R}$ by

$$\mathbf{w}(x, c) = \mathbf{u}(x, f(x) - c) \quad \text{for } (x, c) \in \mathbf{D}$$

We extend the definition of $\mathbf{w}$ to the domain Ω. For $(x, c) \in \Omega$ with $x > 1$ (so that $f(x) = 1$, and $c \le 1$), define

$$\mathbf{w}(x, c) = \mathbf{w}(1, c)$$

Finally, we extend the definition of $\mathbf{w}$ to the domain $\mathscr{R}^2_+$. For (x, c) in $\mathscr{R}^2_+$ with $c > f(x)$, define

$$\mathbf{w}(x, c) = \mathbf{w}(x, f(x))$$

It can be checked for this example that f satisfies (F.1)–(F.3), $\mathbf{w}$ satisfies (W.1)–(W.3).

Furthermore, it can be shown that the optimal policy function $\mathbf{h}:\mathscr{R}_+ \to \mathscr{R}_+$ satisfies

$$\mathbf{h}(x) = 4x(1-x) \quad \text{for } x \in I$$

Referring now to Section 2, we have an example of $(\boldsymbol{f}, \mathbf{w}, \bar{\delta})$ for which the dynamical system $(I, \mathbf{h})$ exhibits topological and ergodic chaos (and also has a positive Lyapunov exponent).

6b. Robustness of topological chaos

In Section 6a, we provided a specification of $(\boldsymbol{f}, \mathbf{w}, \bar{\delta})$ for which topological chaos is seen to occur. A natural question to ask is whether this is fortuitous (so that if the parameters f, w or δ were perturbed ever so little, the property of topological chaos would disappear) or whether this is a "robust" phenomenon (so that small perturbations of f, w or δ would preserve the property of topological chaos).

We proceed now to formalize the above question as follows. Define $\mathscr{F} = \{f:\mathscr{R}_+ \to \mathscr{R}_+ \text{ satisfying (F.1)–(F.3)}\}$; $\mathscr{W} = \{w:\mathscr{R}^3_+ \to \mathscr{R}_+ \text{ satisfying (W.1)–(W.3)}\}$; $\Delta = \{\delta : 0 < \delta < 1\}$. An *economy*, e, is defined by a triple $(f, w, \delta) \in \mathscr{F} \times \mathscr{W} \times \Delta$. The set of economies, $\mathscr{F} \times \mathscr{W} \times \Delta$ is defined by E.

Consider the economy $\mathbf{e} = (\boldsymbol{f}, \mathbf{w}, \bar{\delta})$ defined in Section 6a. We would like to demonstrate that all economies $e \in E$ "near" the economy $\mathbf{e}$ will exhibit topological chaos. Thus, the property of topological chaos will be seen to persist for small perturbations of the original economy.

A convenient way to make the above idea precise is to define for $e \in E$, the "distance" between economies e and $\mathbf{e}$ by

$$d(e, \mathbf{e}) = \sup_{x \geq 0} |f(x) - \boldsymbol{f}(x)| + \sup_{(x,c) \geq 0} |w(x,c) - \mathbf{w}(x,c)| + |\delta - \bar{\delta}|$$

Note that $d(e, \mathbf{e})$ may be infinite.

Before we proceed further, we have to clarify two preliminary points. First, if we perturb the original economy $\mathbf{e}$ (that is, choose another economy $e \neq \mathbf{e}$), we will, in general, change the set of programs from any given initial stock, as also the optimal program from any given initial stock. That is, programs and optimal programs [and hence optimal policy functions] are economy specific. Thus, given an economy e, we use expressions like "e-program", "e-optimal program", "e-optimal policy function" with the obvious meanings.

Second, recall that for the original economy $\mathbf{e}$, $K = 1$, and so if the initial stock was in I, then for any program from the initial stock, the input stock in every period is confined to I. Furthermore, for any initial stock not in I, the input stock on any program belongs to I from the very next period. In this sense, $(I, \mathbf{h})$ is the "natural" dynamical system for the economy $\mathbf{e}$. When we perturb the economy, we do not wish to restrict the *kind* of perturbation in any way, and so we would have to allow the new economy's production function to satisfy (F.3) with a $K > 1$. This changes the "natural" state space of the dynamical system for the new economy. However, recalling that we are only interested in "small" perturbations, it is surely possible to ensue that $d(e, \mathbf{e}) \leq 1$, so that we can legitimately take the "natural" state space choice to be $J = [0, 2]$.

We can now describe our result formally as follows.

Theorem 6.1:
Let $\mathbf{e} = (f, \mathbf{w}, \bar{\delta})$ *be the economy described in Example 6.1. There exists some* $\varepsilon > 0$ *such that for every economy* $e \in E$ *with* $d(e, \mathbf{e}) < \varepsilon$*, the dynamical system* (J, h) *exhibits topological chaos, where h is the e-optimal policy function and* $J = [0, 2]$.

7. Appendix

7.1. Verification of the results of section 4

(a) Verification of Proposition 4.5

Suppose, contrary to the claim, we have $h(x) < f(x)$ for all $0 < x \leq \tilde{x}$. Then we have two cases to consider. Either (i) $h(x) > 0$ for all $x \in (0, \tilde{x}]$, or (ii) $h(x) = 0$ for some $x \in (0, \tilde{x}]$. In case (i), $(x, h(x)) \in \Omega^0$ for $x \in (0, \tilde{x}]$; that is Condition (4.3) is satisfied. But this is inconsistent with (4.2), so case (i) could not arise. In case (ii), noting that $h(\tilde{x}) > 0$, we actually have $h(x) = 0$ for some $x \in (0, \tilde{x})$. Let $x'' = \sup\{x \in (0, \tilde{x}) : h(x) = 0\}$. Then $h(x'') = 0$, and $0 < x'' < \tilde{x}$ by continuity of h. Also, $f(x) > h(x) > 0$ for all $x'' < x < \tilde{x}$, so that $(x, h(x)) \in \Omega^0$ for $x'' < x < \tilde{x}$. By Proposition 4.4, h is non-increasing in $(x'', \tilde{x})$, and the continuity of h leads to $h(x'') \geq h(\tilde{x}) > 0$, a contradiction. Thus case (ii) could not arise. Since these are the only two cases possible, the claim is established.

(b) Verification of Corollary 4.1

We claim that Condition (4.4) implies $h(x) < f(x)$ for all $x > 0$. For if the claim is false, there is some $x > 0$ for which $h(x) = f(x)$. This means that if $\{\bar{x}_t\}_0^\infty$ is the optimal program from x, then $\bar{x}_1 = f(x)$. Consider the sequence $\{x_t'\}_0^\infty$ defined by $x_0' = x$, and $x_t' = (1 - \lambda)x_t$ for $t \geq 0$, where $0 < \lambda < 0.5$. Clearly $\{x_t'\}_0^\infty$ is a program. Now, we have

$$\begin{aligned}\sum_0^\infty \delta^t w(x_t', c_{t+1}') &\geq w(x, \lambda f(x)) + (1 - \lambda)\delta V(f(x)) \\ &= [w(x, \lambda f(x)) - w(x, 0)] + [w(x, 0) + \delta V(f(x))] + [(1 - \lambda)\delta V(f(x)) - \delta V(f(x))] \\ &\geq D_2 w(x, \lambda f(x))\lambda f(x) + V(x) - \lambda\delta V(f(x))\end{aligned}$$

Thus $\sum_0^\infty \delta^t w(x_t', c_{t+1}') - V(x) \geq \lambda[D_2 w(x, \lambda f(x)) - \delta V(f(x))]$. Now $(x, f(x)/2) \in \Omega^0$ and so by Condition (4.4), $D_2 w(x, 2\lambda(f(x)/2)) \to \infty$ as $2\lambda \to 0$. Thus for λ small enough and positive, we can ensure $\lambda[D_2 w(x, \lambda f(x)) - \delta V(f(x))] > 0$. This implies that

$$\sum_0^\infty \delta^t w(x_t', c_{t+1}') > V(x)$$

which is a contradiction to the definition of V. This establishes our claim.

Corollary 4.1 now follows directly from Proposition 4.5.

7.2 Verification of Example 5.1

(a) Verification of Assumptions (F.1)–(F.3), (W.1)–(W.3)

Note that $f(0) = 0$ and $f(x) \to 1$ as $x \to 0.25$ from the left, so f is clearly continuous on $\mathscr{R}_+$ verifying (F.1).

Next, observe that for $0 \leq x < 0.25$, f is C^2 and

$$f'(x) = (32/3) - 64x + (1024/3)x^3$$
$$f''(x) = -64 + 1024x^2$$

As $x \to 0.25$ from the left, $f'(x) \to 0$ and $f''(x) \to 0$, so f is in fact C^2 on $\mathscr{R}_+$. It is clear that $f''(x) < 0$ for $0 \leq x < 0.25$ and so $f'(x) > 0$ for $0 \leq x < 0.25$ [since $f'(0.25) = 0$]. Thus, f is non-decreasing and concave on $\mathscr{R}_+$ verifying (F.2).

Note that (F.3) is satisfied by choosing $K = 1$. For if $x > 1$, then $[f(x)/x] = (1/x) < 1$. If $0.25 \leq x < 1$, $[f(x)/x] = (1/x) > 1$. Also, since $f(0) = 0$ and f is concave, $[f(x)/x]$ is non-increasing in x. Thus, for $0 < x < 0.25$, $[f(x)/x] \geq [f(0.25)/0.25] = 4 > 1$.

The verification of assumptions (W.1)–(W.3) for the given welfare function is too obvious to be spelt out in detail.

(b) Verification of the unique non-trivial stationary optimal stock $x(\delta)$

Since $0.25 < x(\delta) < 1$, we have $f(x(\delta)) = 1 > x(\delta)$, and so $c(\delta) \equiv f(x(\delta)) - x(\delta) = [1/(1+\delta)] > 0$. Define $\{\bar{x}_t\}_0^\infty$ by $\bar{x}_t = x(\delta)$ for $t \geq 0$. Then, $\{\bar{x}_t\}_0^\infty$ is a stationary program from $x(\delta)$, and $\bar{c}_{t+1} = f(\bar{x}_t) - \bar{x}_{t+1} = c(\delta) > 0$ for $t \geq 0$. Next, define a sequence $\{p_t\}_0^\infty$ by $p_t = (\delta^t/\delta^{1/2})$ for $t \geq 0$.

Recalling our definition of Ω and u (see Section 3b), we note that u is C^1 in the interior of Ω (that is, Ω^0), and continuous and concave on Ω. Thus, for any $(x, z) \in \Omega$, and for any $t \geq 0$,

$$u(x,z) - u(\bar{x}_t, \bar{x}_{t+1}) \leq D_1 u(\bar{x}_t, \bar{x}_{t+1})(x - \bar{x}_t) + D_2 u(\bar{x}_t, \bar{x}_{t+1})(z - \bar{x}_{t+1})$$

Now, for any $(x, z) \in \Omega^0$, $x > 0$ and $f(x) - z > 0$. So

$$\left.\begin{aligned} D_2 u(x,z) &= -D_2 w(x, f(x) - z) \\ D_1 u(x,z) &= D_1 w(x, f(x) - z) + f'(x) D_2 w(x, f(x) - z) \end{aligned}\right] \tag{7.1}$$

Since $0.25 < x(\delta) < 1$, $f'(x(\delta)) = 0$, we have $D_2 u(x(\delta), x(\delta)) = -D_2 w(x(\delta), c(\delta))$; $D_1 u(x(\delta), x(\delta)) = D_1 w(x(\delta), c(\delta))$. So,

$$\begin{aligned} u(x, z) - u(\bar{x}_t, \bar{x}_{t+1}) &= D_1 w(x(\delta), c(\delta))(x - \bar{x}_t) - D_2 w(x(\delta), c(\delta))(z - \bar{x}_{t+1}) \\ &= (1/\delta^{1/2})(x - \bar{x}_t) - \delta^{1/2}(z - \bar{x}_{t+1}) \end{aligned}$$

Thus, multiplying through by δ^t, transposing terms, and substituting $p_t = (\delta^t/\delta^{1/2})$, we get

$$\delta^t u(x, z) + p_{t+1} z - p_t x \leq \delta^t u(\bar{x}_t, \bar{x}_{t+1}) + p_{t+1}\bar{x}_{t+1} - p_t \bar{x}_t$$

Also, $p_t \bar{x}_t = p_t x(\delta) \to 0$ as $t \to \infty$. Thus, by Theorem 3.2, $\{\bar{x}_t\}_0^\infty$ is an optimal program from $x(\delta)$, and $x(\delta)$ is a non-trivial stationary optimal stock.

To check that $x(\delta)$ is the only non-trivial stationary optimal stock, we proceed as follows. Suppose $\bar{x}$ is any non-trivial stationary optimal stock. Then, by stationarity, $\bar{x}\in[0,1]$. Since $\bar{x}$ is non-trivial, $\bar{x}\in(0,1]$. By optimality, $\bar{x}\in(0,1)$. Since $f(x)>x$ for all $x\in(0,1)$, $(\bar{x},\bar{x})$ is in Ω^0. Denote $f(\bar{x})-\bar{x}$ by $\bar{c}$; then $\bar{c}>0$.

Now, for each $t\geq 0$, $\bar{x}$ must solve the following problem

$$\begin{aligned}&\text{Max} \quad u(\bar{x},x)+\delta u(x,\bar{x})\\ &\text{Subject to} \quad (\bar{x},x)\in\Omega \text{ and } (x,\bar{x})\in\Omega\end{aligned}$$

Since $(\bar{x},\bar{x})$ is in the interior of Ω, we have the first-order condition

$$D_2u(\bar{x},\bar{x})+\delta D_1u(\bar{x},\bar{x})=0$$

Thus, using (7.1), we get

$$(\bar{x}/\bar{c})^{1/2}=\delta(\bar{x}/\bar{c})^{1/2}f'(\bar{x})+\delta(\bar{c}/\bar{x})^{1/2}$$

Since $f'(\bar{x})\geq 0$, we then have $\bar{x}\geq\delta\bar{c}=\delta[f(\bar{x})-\bar{x}]$. This yields

$$\delta\{[f(\bar{x})/\bar{x}]-1\}\leq 1$$

Since f is concave and $f(0)=0$, $[f(x)/x]$ is non-increasing in x. Thus, for $x\leq 0.25$, $[f(x)/x]\geq[f(0.25)/0.25]=4$, and $\delta[[f(x)/x]-1]\geq 3\delta>1$. Consequently, $\bar{x}>0.25$.

Since $\bar{x}>0.25$, we can return to the above first-order condition, and observe that $f'(\bar{x})=0$; so, we obtain $\bar{x}=\delta\bar{c}=\delta[f(\bar{x})-\bar{x}]=\delta(1-\bar{x})$. This means $\bar{x}=[\delta/(1+\delta)]=x(\delta)$.

(c) Verification of the periodic optimal program

Given any $\tilde{x}\in A(\delta)$, $\tilde{x}\neq x(\delta)$, define

$$\tilde{y}=\delta^2(1-\tilde{x})/[\delta^2(1-\tilde{x})+\tilde{x}]$$

Given the definition of $\tilde{y}$, it is easy to check that $\tilde{y}\neq\tilde{x}$, and

$$[\delta^2(1-\tilde{y})+\tilde{y}][\delta^2(1-\tilde{x})+\tilde{x}]=\delta^2 \tag{7.2}$$

a relation we will find useful in what follows.

We note that $\tilde{y}<\delta^2(1-\tilde{x})/[\delta^2(1-\tilde{x})+0.25]<\delta^2(1-0.25)/[\delta^2(1-0.25)+0.25]=3\delta^2/[1+3\delta^2]$. Also, $\delta^2(1-\tilde{x})/\tilde{x}=\delta^2[(1/\tilde{x})-1]>\delta^2[\{(1+3\delta^2)/3\delta^2\}-1]=(1/3)$. So, $\tilde{y}=\delta^2(1-\tilde{x})/[\delta^2(1-\tilde{x})+\tilde{x}]=1-\{\tilde{x}/[\delta^2(1-\tilde{x})+\tilde{x}]\}=1-(1/[\{\delta^2(1-\tilde{x})/\tilde{x}\}+1])>1-(1/[(1/3)+1])=0.25$. Thus, $\tilde{y}\in A(\delta)$.

Define a sequence $\{\tilde{x}_t\}_0^\infty$ as $\tilde{x}_t=\tilde{x}$ for $t=0,2,4,\ldots$; $\tilde{x}_t=\tilde{y}$ for $t=1,3,5,\ldots$. We can verify that $\{\tilde{x}_t\}_0^\infty$ is a program from $\tilde{x}$. To see this, note that for $t=0,2,4,\ldots,\tilde{x}_{t+1}=\tilde{y}<1$, and $f(\tilde{x}_t)=f(\tilde{x})=1$, so that for $t=0,2,4$,

$$\tilde{c}_{t+1}\equiv f(\tilde{x}_t)-\tilde{x}_{t+1}=\tilde{x}/[\delta^2(1-\tilde{x})+\tilde{x}]$$

Similarly, for $t=1,3,5,\ldots,\tilde{x}_{t+1}=\tilde{x}<1$, and $f(\tilde{x}_t)=f(\tilde{y})=1$, so that for $t=1,3,5,\ldots$

$$\tilde{c}_{t+1}=f(\tilde{x}_t)-\tilde{x}_{t+1}=(1-\tilde{x})$$

It remains to verify that $\{\tilde{x}_t\}_0^\infty$ is an optimal program from $\tilde{x}$. To this end, define a sequence $\{\tilde{p}_t\}_0^\infty$ as follows. For $t = 0, 2, 4, \ldots,$

$$\tilde{p}_{t+1} = \delta^t[\delta^2(1-\tilde{x}) + \tilde{x}]^{1/2} \tag{7.3}$$

For $t = 1, 3, 5, \ldots,$

$$\tilde{p}_{t+1} = \delta^{t+1}/[\delta^2(1-\tilde{x}) + \tilde{x}]^{1/2} \tag{7.4}$$

Further, define

$$\tilde{p}_0 = 1/[\delta^2(1-\tilde{x}) + \tilde{x}]^{1/2} \tag{7.5}$$

Since $(\tilde{x}_t, \tilde{x}_{t+1}) \in \Omega^0$ for $t \geq 0$, we obtain, for any $(x, z) \in \Omega$, and $t \geq 0$, [using (7.1)]

$$\begin{aligned} u(x,z) - u(\tilde{x}_t, \tilde{x}_{t+1}) &\leq D_1 u(\tilde{x}_t, \tilde{x}_{t+1})(x - \tilde{x}_t) + D_2 u(\tilde{x}_t, \tilde{x}_{t+1})(z - \tilde{x}_{t+1}) \\ &= D_1 w(\tilde{x}_t, \tilde{c}_{t+1})(x - \tilde{x}_t) - D_2 w(\tilde{x}_t, \tilde{c}_{t+1})(z - \tilde{x}_{t+1}) \end{aligned}$$

For $t = 0, 2, 4, \ldots,$ we then have

$$u(x,z) - u(\tilde{x}_t, \tilde{x}_{t+1}) \leq \{1/[\delta^2(1-\tilde{x}) + \tilde{x}]\}^{1/2}(x - \tilde{x}) - [\delta^2(1-\tilde{x}) + \tilde{x}]^{1/2}(z - \tilde{y}).$$

So, multiplying through by δ^t, and using (7.3)–(7.5),

$$\delta^t[u(x,z) - u(\tilde{x}_t, \tilde{x}_{t+1})] \leq \tilde{p}_t(x - \tilde{x}) - \tilde{p}_{t+1}(z - \tilde{y}).$$

Transposing terms, for any $(x, z) \in \Omega$, and $t = 0, 2, 4, \ldots,$

$$\delta^t u(\tilde{x}_t, \tilde{x}_{t+1}) + \tilde{p}_{t+1}\tilde{x}_{t+1} - \tilde{p}_t \tilde{x}_t \geq \delta^t u(x,z) + \tilde{p}_{t+1} z - \tilde{p}_t x \tag{7.6}$$

For $t = 1, 3, 5, \ldots,$ we have

$$\begin{aligned} u(x,z) - u(\tilde{x}_t, \tilde{x}_{t+1}) &\leq \{1/[\delta^2(1-\tilde{y}) + \tilde{y}]\}^{1/2}(x - \tilde{y}) - [\delta^2(1-\tilde{y}) + \tilde{y}]^{1/2}(z - \tilde{x}) \\ &= (1/\delta)[\delta^2(1-\tilde{x}) + \tilde{x}]^{1/2}(x - \tilde{y}) - \{\delta/[\delta^2(1-\tilde{x}) + \tilde{x}]\}^{1/2}(z - \tilde{x}) \end{aligned}$$

using the relation (7.2). Thus, multiplying through by δ^t, we have

$$\delta^t[u(x,z) - u(\tilde{x}_t, \tilde{x}_{t+1})] \leq \tilde{p}_t(x - \tilde{y}) - \tilde{p}_{t+1}(z - \tilde{x})$$

using (7.3) and (7.4). Transposing terms, we obtain, for any $(x, z) \in \Omega$, and $t = 1, 3, 5, \ldots$

$$\delta^t u(\tilde{x}_t, \tilde{x}_{t+1}) + \tilde{p}_{t+1}\tilde{x}_{t+1} - \tilde{p}_t \tilde{x}_t \geq \delta^t u(x,z) + \tilde{p}_{t+1} z - \tilde{p}_t x \tag{7.7}$$

Note finally that $\tilde{p}_t \to 0$ as $t \to \infty$ and $\tilde{x}_t$ is either $\tilde{x}$ or $\tilde{y}$ for each $t \geq 0$. Hence, we have

$$\lim_{t \to \infty} \tilde{p}_t \tilde{x}_t = 0 \tag{7.8}$$

Using (7.6)–(7.8) and Theorem 3.2, $\{\tilde{x}_t\}_0^\infty$ is an optimal program from $\bar{x}$.

7.3 Verification of Example 6.1

(a) Verification of Assumptions (F.1)–(F.3), (W.1)–(W.3)

Note that $f(0) = 0$, and as $x \to 0.5$ from the left, $f(x) \to 1$. So f is clearly continuous on $\mathscr{R}_+$, verifying (F.1).

Next, note that for $0 \le x < 0.5$, f is C^2, and

$$f'(x) = (16/3) - 16x + (64/3)x^2$$
$$f''(x) = -16 + 64x^2$$

As $x \to 0.5$ from the left, $f'(x) \to 0$ and $f''(x) \to 0$, so f is in fact C^2 on $\mathscr{R}_+$. It is clear that $f''(x) < 0$ for $0 \le x < 0.5$, and so $f'(x) > 0$ for $0 \le x < 0.5$ [since $f'(0.5) = 0$]. Thus, f is non-decreasing and concave on $\mathscr{R}_+$, verifying (F.2).

Note that (F.3) is satisfied by choosing $K = 1$. For if $x > 1$, then $[f(x)/x] = (1/x) < 1$. If $0.5 \le x < 1$, $[f(x)/x] = (1/x) > 1$. Also, since $f(0) = 0$, and f is concave, $[f(x)/x]$ is non-increasing in x. Thus, for $0 < x < 0.5$, $[f(x)/x] \ge [f(0.5)/0.5] = 2 > 1$.

Before proceeding to verify the assumptions (W.1)–(W.3), we note an important property of f, namely

$$f(x) \ge \theta(x) \quad \text{for } x \in I$$

To see this, note that $f(x) = 1$ for $0.5 \le x \le 1$, and $\theta(x) \in I$ for all $x \in I$ so $f(x) \ge \theta(x)$ for $0.5 \le x \le 1$. Also, $f(0) = \theta(0) = 0$. For $0 < x < 0.5$, we define $\beta(x) \equiv [f(x) - \theta(x)]/x$. Then $\beta(x) = (4/3) - 4x + (16/3)x^3$, and $\beta'(x) = -4 + 16x^2$. Thus $\beta'(x) < 0$ for $0 < x < 0.5$, and β is a decreasing function. As $x \to 0.5$ from the left $\beta(x) \to 0$. Thus, $\beta(x) > 0$ for $0 < x < 0.5$, and so $f(x) > \theta(x)$ for $0 < x < 0.5$.

We now verify (W.1)–(W.3). Recalling the definition of $\mathbf{u}: I^2 \to \mathscr{R}$, we can compute the following derivatives:

$$D_1\mathbf{u}(x,z) = a - Lx + 4z(1-2x)$$
$$D_2\mathbf{u}(x,z) = 4x(1-x) - z - \bar{\delta}a + \bar{\delta}Lz - 16\bar{\delta}z + 48\bar{\delta}z^2 - 32\bar{\delta}z^3$$
$$D_{11}\mathbf{u}(x,z) = -L - 8z$$
$$D_{12}\mathbf{u}(x,z) = 4(1-2x) = D_{21}\mathbf{u}(x,z)$$
$$D_{22}\mathbf{u}(x,z) = -1 + \bar{\delta}L - 16\bar{\delta} + 96\bar{\delta}z - 96\bar{\delta}z^2$$

We first check (W.1)–(W.3) on the set $\mathbf{D}$. Clearly, $\mathbf{w}$ is C^2 on $\mathbf{D}$, and we can compute the first partials of $\mathbf{w}$ as follows:

$$D_2\mathbf{w}(x,c) = -D_2\mathbf{u}(x, f(x) - c)$$
$$D_1\mathbf{w}(x,c) = D_1\mathbf{u}(x, f(x) - c) + D_2\mathbf{u}(x, f(x) - c)f'(x)$$

Now,

$$D_2\mathbf{u}(x, f(x) - c) \le 1 - \bar{\delta}a + \bar{\delta}z(L-16) - z + 48\bar{\delta} \le 1 - \bar{\delta}a + 48\bar{\delta} \quad [\text{since } \bar{\delta}(L-16) \le 1]$$
$$\le 1 - \bar{\delta}(a - 48)$$
$$< 0 \ [\text{since } \bar{\delta}(a-48) > 1]$$

Thus $D_2\mathbf{w}(x,c) > 0$ for $(x,c) \in \mathbf{D}$. Also, $D_1\mathbf{u}(x,z) \ge a - L - 4 \ge 323$, $f'(x) \le f'(0) = (16/3)$, and

$$D_2\mathbf{u}(x,z) \ge -\bar{\delta}a - z + \bar{\delta}z(L-16) + 16\bar{\delta}z^2(3-2z) \ge -\bar{\delta}a - z \ [\text{since } L > 16]$$
$$\ge -2.0625$$

Thus, $D_2\mathbf{u}(x,z)f'(x) \geq -16$, and $D_1\mathbf{w}(x,c) \geq 323 - 16 > 0$. Thus, $\mathbf{w}$ is strictly increasing in c (given x) and in x (given c) on $\mathbf{D}$.

The second partials of $\mathbf{w}$ are:

$$D_{22}\mathbf{w}(x,c) = D_{22}\mathbf{u}(x, f(x) - c)$$
$$D_{21}\mathbf{w}(x,c) = -D_{21}\mathbf{u}(x, f(x) - c) - D_{22}\mathbf{u}(x, f(x) - c)f'(x) = D_{12}\mathbf{w}(x,c)$$
$$D_{11}\mathbf{w}(x,c) = D_{11}\mathbf{u}(x, f(x) - c) + D_{12}\mathbf{u}(x, f(x) - c)f'(x) + [D_{21}\mathbf{u}(x, f(x) - c) + D_{22}\mathbf{u}(x, f(x) - c)f'(x)]f'(x) + D_2\mathbf{u}(x, f(x) - c)f''(x)$$

Now, $D_{22}\mathbf{u}(x,z) \leq -1 + \bar{\delta}L + 96\bar{\delta}z \leq -1 + \bar{\delta}(L + 96) < -0.5 < 0$; and $D_{22}\mathbf{u}(x,z) \geq -1 + \bar{\delta}(L - 16) > -1$. Also, $D_{11}\mathbf{u}(x,z) \leq -L < 0$; and $|D_{12}\mathbf{u}(x,z)| \leq 4$. These estimates imply that

$$(D_{11}\mathbf{u})(D_{22}\mathbf{u}) - (D_{12}\mathbf{u})^2 > (L/2) - 16 = 33 > 0$$

Also, $D_2\mathbf{u}(x,z) > -2.0625$ and $D_{22}\mathbf{u}(x,z) > -1$ as checked above; while $f''(x) = -16 + 64x^2$ for $0 \leq x \leq 0.5$, so $0 \geq f''(x) \geq -16$. Thus, $(D_{22}\mathbf{u})(D_2\mathbf{u})f''(x) > -33$.

Now, $(D_{11}\mathbf{w})(D_{22}\mathbf{w}) - (D_{12}\mathbf{w})^2 = (D_{22}\mathbf{u})(D_{11}\mathbf{u}) + 2(D_{12}\mathbf{u})(D_{22}\mathbf{u})f'(x) + (D_{22}\mathbf{u})^2 f'(x)^2 + (D_{22}\mathbf{u})(D_2\mathbf{u})f''(x) - (D_{12}\mathbf{u})^2 - (D_{22}\mathbf{u})^2 f'(x)^2 - 2(D_{12}\mathbf{u})(D_{22}\mathbf{u})f'(x) = (D_{22}\mathbf{u})(D_{11}\mathbf{u}) - (D_{12}\mathbf{u})^2 + (D_{22}\mathbf{u})(D_2\mathbf{u})f''(x)$. Thus, using the above estimates, we have

$$D_{11}\mathbf{w}(x,c)D_{22}\mathbf{w}(x,c) - (D_{12}\mathbf{w}(x,c))^2 > 33 - 33 = 0$$

Also, $D_{22}\mathbf{w}(x,c) = D_{22}\mathbf{u}(x, f(x) - c) < 0$, so that $\mathbf{w}(x,c)$ is strictly concave on the set $\mathbf{D}$.

Next, we check that $\mathbf{w}$ satisfies (W.1)–(W.3) on Ω. By definition of $\mathbf{w}$ on Ω, it satisfies (W.1) and (W.2). To check (W.3), let (x,c) and $(\bar{x},\bar{c})$ belong to Ω and let $0 < \lambda < 1$. Then $\mathbf{w}(\lambda(x,c) + (1-\lambda)(\bar{x},\bar{c})) = \mathbf{w}(\lambda x + (1-\lambda)\bar{x},\ \lambda c + (1-\lambda)\bar{c})$. Now $[\lambda x + (1-\lambda)\bar{x},\ \lambda c + (1-\lambda)\bar{c}]$ is in $\mathbf{D}$ if $[\lambda x + (1-\lambda)\bar{x}] \leq 1$. If x and $\bar{x}$ are both ≤ 1, then concavity of $\mathbf{w}$ follows from concavity of $\mathbf{w}$ on $\mathbf{D}$. So, consider without loss of generality that $x > 1$ while $\bar{x} < 1$. Then $\mathbf{w}(\lambda x + (1-\lambda)\bar{x}, \lambda c + (1-\lambda)\bar{c}) \geq \mathbf{w}(\lambda + (1-\lambda)\bar{x}, \lambda c + (1-\lambda)\bar{c})$ [using (W.2)] $\geq \lambda\mathbf{w}(1,c) + (1-\lambda)\mathbf{w}(\bar{x},\bar{c})$ [since $(1,c)$ and $(\bar{x},\bar{c})$ belong to $\mathbf{D}$] $= \lambda\mathbf{w}(x,c) + (1-\lambda)\mathbf{w}(\bar{x},\bar{c})$. If $[\lambda x + (1-\lambda)\bar{x}] > 1$, then $\mathbf{w}(\lambda x + (1-\lambda)\bar{\mathbf{x}},\ \lambda c + (1-\lambda)\bar{c}) = \mathbf{w}(1, \lambda c + (1-\lambda)\bar{c}) \geq \lambda\mathbf{w}(1,c) + (1-\lambda)\mathbf{w}(1,\bar{c})$ [since $(1,c)$ and $(1,\bar{c})$ belong to D] $\geq \lambda\mathbf{w}(x,c) + (1-\lambda)\mathbf{w}(\bar{x},\bar{c})$.

If $0 \leq x \leq 1$ is fixed, then $\mathbf{w}(x,c)$ is strictly concave in c [since $D_{22}w(x,c) < 0$ on $\mathbf{D}$]; If $x > 1$ is fixed, $\mathbf{w}(x,c) = \mathbf{w}(1,c)$ is strictly concave in c [since $D_{22}w(x,c) < 0$ on $\mathbf{D}$].

Finally, we check that $\mathbf{w}$ satisfies (W.1)–(W.3) on $\mathscr{R}^2_+$. Again, $\mathbf{w}$ clearly satisfies (W.1) and (W.2). To check (W.3), let (x,c) and $(\bar{x},\bar{c})$ belong to $\mathscr{R}^2_+$ and let $0 < \lambda < 1$. Denote $\min[c, f(x)]$ by $G(x,c)$; $\min[\bar{c}, f(\bar{x})]$ by $G(\bar{x},\bar{c})$. Then, we have $\mathbf{w}(\lambda x + (1-\lambda)\bar{x},\ \lambda c + (1-\lambda)\bar{c}) \geq \mathbf{w}(\lambda x + (1-\lambda))\bar{x},\ \lambda G(x,c) + (1-\lambda)G(\bar{x},\bar{c}))$. Also, $\lambda G(x,c) + (1-\lambda)G(\bar{x},\bar{c}) \leq \lambda f(x) + (1-\lambda)f(\bar{x}) \leq f(\lambda x + (1-\lambda)\bar{x})$. Further, $G(x,c) \leq f(x)$ and $G(\bar{x},\bar{c}) \leq f(\bar{x})$. Thus, $(x, G(x,c))$, $(\bar{x}, G(\bar{x},\bar{c}))$ and $[\lambda x + (1-\lambda)\bar{x},\ \lambda G(x,c) + (1-\lambda)G(\bar{x},\bar{c})]$ all belong to Ω. Thus, $\mathbf{w}(\lambda x + (1-\lambda)\bar{x}, \lambda c + (1-\lambda)\bar{c}) \geq \lambda\mathbf{w}(x, G(x,c)) + (1-\lambda)\mathbf{w}(\bar{x}, G(\bar{x},\bar{c}))$. If $\min[c, f(x)] = c$ then $\mathbf{w}(x, G(x,c)) = \mathbf{w}(x,c)$; if $\min[c, f(x)] \neq c$, then $c > f(x)$, and $\mathbf{w}(x, G(x,c)) = \mathbf{w}(x, f(x)) = \mathbf{w}(x,c)$; thus, in either case, $\mathbf{w}(x, G(x,c)) = \mathbf{w}(x,c)$. Similarly, $\mathbf{w}(\bar{x}, G(\bar{x},\bar{c})) = \mathbf{w}(\bar{x},\bar{c})$. Hence, $\lambda\mathbf{w}(x, G(x,c)) + (1-\lambda)\mathbf{w}(\bar{x}, G(\bar{x},\bar{c})) = \lambda\mathbf{w}(x,c) + (1-\lambda)\mathbf{w}(\bar{x},\bar{c})$, completing our demonstration of the concavity of $\mathbf{w}$ on $\mathscr{R}^2_+$.

It is worth noting that by (W.1), $\mathbf{w}(x,c) \geq \mathbf{w}(0,0) = \mathbf{u}(0, f(0) - 0) = 0$ for all $(x,c) \in \mathscr{R}_+^2$. Thus $\mathbf{w}$ maps $\mathscr{R}_+^2$ to $\mathscr{R}_+$. Also, for all $(x,c) \in \mathscr{R}_+^2$, $\mathbf{w}(x,c) \leq \mathbf{w}(1,c) \leq \mathbf{w}(1, f(1)) = \mathbf{w}(1,1) = \mathbf{u}(1,0) = 376$.

(b) Verification of the optimal policy function

This part of our verification relies heavily on the technique of Boldrin-Montruchhio (1986b).

Define $\phi: I^2 \to \mathscr{R}$ by

$$\phi(x,z) = ax - 0.5Lx^2 + z\theta(x) - 0.5z^2$$

Clearly ϕ is C^2 on I^2, and we can compute the following derivatives:

$$\begin{aligned} D_1\phi(x,z) &= a - Lx + z\theta'(x) \\ D_2\phi(x,z) &= \theta(x) - z \\ D_{11}\phi(x,z) &= -L + z\theta''(x) \\ D_{12}\phi(x,z) &= \theta'(x) = D_{21}\phi(x,z) \\ D_{22}\phi(x,z) &= -1 \end{aligned}$$

Now $\theta'(x) = 4 - 8x$ and $\theta''(x) = -8$. Thus $|\theta'(x)| \leq 4$ for $x \in I$. Also, looking at the Hessian matrix of ϕ, we note that $D_{11}\phi(x,z) < 0$ since $L > 0$ and $\theta''(x) < 0$. And $(D_{11}\phi)(D_{22}\phi) - (D_{12}\phi)^2 = L - z\theta''(x) - [\theta'(x)]^2 \geq L - 16 > 0$. Thus ϕ is strictly concave on I^2.

Clearly $\theta(x) \in I$ for all $x \in I$. And, for any $z \in I$, $z \neq \theta(x)$, we have by the strict concavity of ϕ,

$$\phi(x,z) - \phi(x,\theta(x)) < D_2\phi(x,\theta(x))(z - \theta(x)) = 0$$

Hence, for every $x \in I$, $\theta(x)$ uniquely solves the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Max} & \phi(x,z) \\ \text{Subject to} & z \in I \end{array}\right\} \qquad \text{(P)}$$

We define $\psi: I \to I$ by

$$\psi(x) = \phi(x,\theta(x)) \quad \text{for } x \in I$$

Then by definitions of ϕ and θ, we have

$$\psi(x) = ax - 0.5Lx^2 + \theta(x)^2 - 0.5\theta(x)^2 = ax - 0.5Lx^2 + 0.5\theta(x)^2$$

Clearly ψ is a continuous function on I.

Now, note that for $(x,z) \in I^2$, the definitions of ϕ, ψ and $\mathbf{u}$ yields the equation

$$\mathbf{u}(x,z) = \phi(x,z) - \bar{\delta}\psi(z)$$

Thus for every $x \in I$, $\theta(x)$ uniquely solves the problem:

$$\left.\begin{array}{ll} \text{Max} & \mathbf{u}(x,z) + \bar{\delta}\psi(z) \\ \text{Subject to} & z \in I \end{array}\right\} \qquad \text{(Q)}$$

But since $\theta(x) \leq f(x)$ for $x \in I$, we can clearly also conclude that $\theta(x)$ uniquely solves the problem

$$\left.\begin{array}{ll} \text{Max} & \mathbf{u}(x,z) + \bar{\delta}\psi(z) \\ \text{Subject to} & (x,z) \in \Omega \end{array}\right\} \qquad (\mathrm{Q}')$$

This means that for all $x \in I$,

$$\operatorname*{Max}_{(x,z)\in\Omega} [\mathbf{u}(x,z) + \bar{\delta}\psi(z)] = \mathbf{u}(x,\theta(x)) + \bar{\delta}\psi(\theta(x)) = \phi(x,\theta(x)) = \psi(x)$$

Applying Theorem 3.1 (i), we can conclude that $\psi(x) = \mathbf{V}(x)$ for all $x \in I$. Thus $\theta(x)$ uniquely solves the problem:

$$\left.\begin{array}{ll} \text{Max} & \mathbf{u}(x,z) + \bar{\delta}\mathbf{V}(z) \\ \text{Subject to} & (x,z) \in \Omega \end{array}\right\} \qquad (\mathrm{Q}'')$$

Applying Theorem 3.1 (ii), $\theta(x) = \mathbf{h}(x)$ for all $x \in I$.

7.4 Verification of Theorem 6.1

We describe our strategy of proof briefly before providing the formal details. First, we consider an **e**-optimal program from a *specific* initial input [we actually choose $\mathbf{x}_0 = (1/8)$] for which

$$h^3(\mathbf{x}_0) < \mathbf{x}_0 < \mathbf{h}(\mathbf{x}_0) < \mathbf{h}^2(\mathbf{x}_0)$$

This enables us to find a number $\eta > 0$ [we actually have $\eta = (1/32)$] such that if $\{x'_t\}_0^\infty$ is any sequence with $x'_0 = \mathbf{x}$ and $|x'_t - \mathbf{x}_t| \leq \eta$ for $t = 1, 2, 3$, then

$$x'_3 + \eta < x'_0 < x'_1 - \eta < x'_2 - 2\eta$$

Thus, it is enough to show that for an economy e near **e**, the e-optimal program from **x** stays within η of the **e**-optimal program from **x** *in the first three periods* to invoke the Li-Yorke theorem (Theorem 2.1) and conclude robustness of topological chaos. This appears to be intuitively plausible, but it does involve checking the details that we provide below.

We proceed formally as follows. Let $\{\mathbf{x}_t\}_0^\infty$ be the **e**-optimal program from $\mathbf{x} = (1/8)$. Then $\mathbf{x}_0 = (1/8)$, $\mathbf{x}_1 = \mathbf{h}(\mathbf{x}_0) = (7/16)$, $\mathbf{x}_2 = \mathbf{h}^2(\mathbf{x}_0) = (63/64)$ and $\mathbf{x}_3 = \mathbf{h}^3(\mathbf{x}_0) = (63/1024)$. Clearly, we have

$$\mathbf{x}_3 < \mathbf{x}_0 < \mathbf{x}_1 < \mathbf{x}_2$$

We now break up our proof into several steps

Step 1: Let $\{x'_t\}_0^\infty$ *be any sequence with* $x'_0 = \mathbf{x} = (1/8)$. *If* $\max_{t=1,2,3} |x'_t - \mathbf{x}_t| \leq (1/32)$, then

$$x'_3 + (1/32) < x'_0 < x'_1 - (1/32) < x'_2 - (1/16)$$

To see Step 1, note first that $x'_1 - x'_0 = x'_1 - \mathbf{x} = (x'_1 - \mathbf{x}_1) + (\mathbf{x}_1 - \mathbf{x}) \geq (\mathbf{x}_1 - \mathbf{x}) - (1/32) = (5/16) - (1/32) = (9/32) > (1/32)$. Second, $x'_2 - x'_1 = (x'_2 - \mathbf{x}_2) + (\mathbf{x}_2 - \mathbf{x}_1) + (\mathbf{x}_1 - x'_1) \geq (\mathbf{x}_2 - \mathbf{x}_1) - (1/32) - (1/32) = (35/64) - (4/64) = (31/64) > (1/32)$. Finally, $x'_0 - x'_3 = \mathbf{x} - x'_3 = (\mathbf{x} - \mathbf{x}_3) + (\mathbf{x}_3 - x'_3) \geq (\mathbf{x} - \mathbf{x}_3) - (1/32) > (1/8) - (1/16) - (1/32) = (1/32)$.

Define a set of programs, Y as follows:

$$Y=\{\{x_t\}_0^\infty:\{x_t\}_0^\infty \text{ is an } \mathbf{e}\text{-program from } \mathbf{x}=(1/8), \text{ such that } \max_{t=1,2,3}|x_t-\mathbf{x}_t|\geq(1/32)\}$$

Step 2: There is $\beta>0$ such that if $\{x_t'\}_0^\infty$ is any **e**-*program belonging to Y, then*

$$\sum_0^\infty \bar{\delta}^t\mathbf{w}(x_t',c_{t+1}')\leq\mathbf{V}(\mathbf{x})-\beta \tag{7.9}$$

To verify this claim, suppose on the contrary for each integer $n\geq1$, there is an e-program $\{x_t^n\}_0^\infty$ in Y satisfying

$$\sum_0^\infty \bar{\delta}^t\mathbf{w}(x_t^n,c_{t+1}^n)\geq\mathbf{V}(\mathbf{x})-(1/n)$$

We can then pick a subsequence (retain notation) such that for each $t\geq1$

$$x_t^n\to\bar{x}_t \quad \text{as } n\to\infty$$

Clearly $\{\bar{x}_t\}_0^\infty$ is an **e**-program which belongs to Y. And, since $\{\mathbf{x}_t\}_0^\infty$ is the unique e-optimal program from $\mathbf{x}$, there is $\alpha>0$ such that

$$\sum_0^\infty \bar{\delta}^t\mathbf{w}(\bar{x}_t,\bar{c}_{t+1})\leq\mathbf{V}(\mathbf{x})-\alpha$$

Given α, we can pick a positive integer T large enough so that $\bar{\delta}^TM_1(1-\bar{\delta})<(\alpha/4)$, where $M_1\equiv\mathbf{w}(1,1)>0$. Then we can pick N large enough so that $n\geq N$ implies that for all $t\in[0,1,\ldots,T]$,

$$|\mathbf{w}(x_t^n,c_{t+1}^n)-\mathbf{w}(\bar{x}_t,\bar{c}_{t+1})|<[\alpha(1-\bar{\delta})/4]$$

Then, for $n\geq N$,

$$\begin{aligned}\sum_0^\infty \bar{\delta}^t\mathbf{w}(\bar{x}_t,\bar{c}_{t+1})&\geq\sum_0^T \bar{\delta}^t\mathbf{w}(\bar{x},\bar{c}_{t+1})\\&\geq\sum_0^T \bar{\delta}^t\mathbf{w}(x_t^n,c_{t+1}^n)-(\alpha/4)\\&\geq\sum_0^\infty \bar{\delta}^t\mathbf{w}(x_t^n,c_{t+1}^n)-(\alpha/2)\\&\geq\mathbf{V}(\mathbf{x})-(1/n)-(\alpha/2)\end{aligned} \tag{7.10}$$

Combining (7.9) and (7.10),

$$\mathbf{V}(\mathbf{x})-\alpha\geq\mathbf{V}(\mathbf{x})-(\alpha/2)-(1/n)$$

Thus $(1/n)\geq(\alpha/2)$ for all $n\geq N$, a contradiction which establishes the claim.

We now introduce some notation for the next (and crucial) step in the proof. Define $\hat{a}=f(\mathbf{x})-\mathbf{x}$. Clearly $\hat{a}>0$. Note that $2\bar{\delta}<1$, and define $M_2=[1/(1-2\bar{\delta})]$. Now choose $0<\lambda<1$ with λ sufficiently close to 1 so that

$$2(1-\lambda)M_1M_2\leq(\beta/4) \tag{7.11}$$

where β is given in Step 2. Next, define $M_3 = \sum_0^\infty (t+1)(2\bar{\delta})^t$. Note that $\mathbf{w}$ is C^1 on D, and define $M_4 = \max_{(x,c)\in D} D_2\mathbf{w}(x,c)$. Finally define

$$\varepsilon = \min\,[(1-\lambda)8, 0.00125, M_1, \{\beta/4[2M_1M_3 + 2M_2 + M_2M_4]\}] \qquad (7.12)$$

Step 3: Consider any $e \in E$ with $d(e,\mathbf{e}) < \varepsilon$, where ε is defined by (7.12). Let $\{x'_t\}_0^\infty$ be the e-optimal program from $\mathbf{x} = (1/8)$. Consider the sequence $\{x''_t\}_0^\infty$ defined by $x''_t = \lambda x'_t + (1-\lambda)\mathbf{x}$ for $t \geq 0$, where $0 < \lambda < 1$ satisfies (7.11). Then $\{x''_t\}_0^\infty$ is an e-program from $\mathbf{x}$ which does not belong to Y.

Note that for $e \in E$ with $d(e,\mathbf{e}) < \varepsilon$, $f(x) \leq \mathbf{f}(x) + \varepsilon$ for all $x \geq 0$, so that $f(x) \leq 1 + \varepsilon$ for $x \geq 0$. This means that $K \leq (1+\varepsilon) < 2$. Thus, defining $J = [0, 2]$ we certainly have $x_t \in J$ for all $t \geq 0$, for every e-program $\{x_t\}_0^\infty$ from every initial stock in J.

Using the definition of $\{x''_t\}_0^\infty$ it is routine to check that $\{x''_t\}_0^\infty$ is an e-program as well as an e-program from x.

Define another sequence $\{\bar{x}_t\}_0^\infty$ by $\bar{x}_t = \lambda \mathbf{x}_t + (1-\lambda)\mathbf{x}$ for $t \geq 0$. Then again it is straightforward to check that $\{\bar{x}_t\}_0^\infty$ is an e-program as well as an e-program from x.

We have to show that $\{x''_t\}_0^\infty$ does not belong to Y. Suppose it did. Then by Step 2 we would have

$$\sum_0^\infty \bar{\delta}^t \mathbf{w}(x''_t, \mathbf{f}(x''_t) - x''_{t+1}) \leq \mathbf{V}(\mathbf{x}) - \beta \qquad (7.13)$$

Now using $d(e,\mathbf{e}) < \varepsilon$, and the fact that $\{\bar{x}_t\}_0^\infty$ is close to $\{\mathbf{x}_t\}_0^\infty$, one can obtain the following estimate

$$\sum_0^\infty \delta^t w(\bar{x}_t, f(\bar{x}_t) - \bar{x}_{t+1}) \geq \mathbf{V}(\mathbf{x}) - \varepsilon M_1 M_3 - \varepsilon M_2 - (1-\lambda) M_1 M_2 \qquad (7.14)$$

Comparing $\{x''_t\}_0^\infty$ as an e-program and as an e-program, and using $d(e,\mathbf{e}) < \varepsilon$, we can also obtain the estimate

$$\sum_0^\infty \delta^t w(x''_t, f(x''_t) - x''_{t+1}) \leq \sum_0^\infty \bar{\delta}^t \mathbf{w}(x''_t, \mathbf{f}(x''_t) - x''_{t+1}) + \varepsilon[M_1M_3 + M_2 + M_2M_4] \qquad (7.15)$$

Combining (7.13)–(7.15),

$$\sum_0^\infty \delta^t w(\bar{x}_t, f(\bar{x}_t) - \bar{x}_{t+1}) \geq \sum_0^\infty \delta^t w(x'_t, f(x'_t) - x'_{t+1}) + (\beta/4)$$

which contradicts the fact that $\{x'_t\}_0^\infty$ is e-optimal from x.

Step 4: Consider any $e \in E$, with $d(e,\mathbf{e}) < \varepsilon$ where ε is defined by (7.12). Let h be the e-optimal policy function. Then

$$h^3(\mathbf{x}) < \mathbf{x} < h(\mathbf{x}) < h^2(\mathbf{x}) \qquad (7.16)$$

To see Step 4, consider the sequences $\{x'_t\}_0^\infty$ and $\{x''_t\}_0^\infty$ defined in Step 3. Since we concluded that $\{x''_t\}_0^\infty$ is an **e**-program which does *not* belong to Y, we must have $\max_{t=1,2,3} |x''_t - \mathbf{x}_t| < (1/32)$. Thus, by Step 1, we have

$$x''_3 + (1/32) < x''_0 < x''_1 - (1/32) < x''_2 - (1/16) \tag{7.17}$$

Now for any s, $t \geq 0$, $(x''_s - x''_t) = [\lambda x'_s + (1-\lambda)\mathbf{x}] - [\lambda x'_t + (1-\lambda)\mathbf{x}] = \lambda(x'_s - x'_t)$. Thus, using, in turn, $s = 1, t = 0$; $s = 2, t = 1$; and $s = 0, t = 3$; and (7.17), we get

$$x'_3 < x'_0 < x'_1 < x'_2 \tag{7.18}$$

Since h is the e-optimal policy function, and $\{x'_t\}_0^\infty$ the e-optimal program from $\mathbf{x}$, (7.18) yields (7.16).

We can now apply the Li-Yorke theorem (Theorem 2.1) to the dynamical system (J, h) where $J = [0, 2]$ and h is the e-optimal policy function for $e \in E$ satisfying $d(e, \mathbf{e}) < \varepsilon$ [and ε is given by (7.12)]. Given (7.16), (J, h) exhibits topological chaos, which establishes Theorem 6.1.

References

Arrow, K. J., Kurz, M.: Public investment, the rate of return and optimal fiscal policy. Baltimore: Johns Hopkins Press 1970

Baumol, W. J., Benhabib, J.: Chaos: significance, mechanism and economic applications. J. Econ. Perspectives **3**, 77–105 (1989)

Benhabib, J., Nishimura, N.: Competitive equilibrium cycles. J. Econ. Theory **35**, 284–306 (1985)

Boldrin, M., Montrucchio, L.: Cyclic and chaotic behavior in intertemporal optimization models. Math. Modelling **8**, 697–700 (1986a)

Boldrin, M., Montrucchio, L.: On the indeterminacy of capital accumulation paths. J. Econ. Theory **40**, 26–39 (1986b)

Boldrin, M., Woodford, M.: Equilibrium models displaying endogenous fluctuations and chaos: a survey. J. Monetary Econ. **25**, 189–222 (1990)

Brock, W. A., Dechert, W. D.: Nonlinear dynamical systems: instability and chaos in economics. In: Hildenbrand, W., Sonnenschein, H. (eds.). Handbook of mathematical economics, vol. IV. Amsterdam: North Holland 1991

Clark, C. W.: Mathematical bioeconomics. New York: Wiley 1976

Collet, P., Eckmann, J. P.: Iterated maps on the interval as dynamical systems. Boston: Birkhäuser 1980

Dasgupta, P. A.: The control of resources. Oxford: Blackwell 1982

Day, R. H., Pianigiani, G.: Statistical dynamics and economics. J. Econ. Behav. Organiz. **16**, 37–84 (1991)

Day, R. H., Shafer, W.: Ergodic fluctuations in economic models. J. Econ. Behav. Organiz. **8**, 339–361 (1987)

Dechert, W. D., Nishimura, K.: A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. J. Econ. Theory **31**, 332–354 (1983)

Deneckere, R., Pelikan, S.: Competitive chaos. J. Econ. Theory **40**, 13–25 (1986)

Devaney, R. L.: An introduction to chaotic dynamic systems. New York: Addison-Welsey 1987

Dutta, P. K., Mitra, T.: Maximum theorems for convex structures with an application to the theory of optimal intertemporal allocation. J. Math. Econ. **18**, 77–86 (1989)

Gale, D.: On optimal development in a multi-sector economy. Rev. Econ. Stud. **34**, 1–18 (1967)

Grandmont, J. M.: On endogenous competitive business cycles. Econ. **53**, 995–1045 (1985)

Grandmont, J. M.: Periodic and aperiodic behavior in discrete one dimensional dynamical systems. In: Hildenbrand, W., Mas-Colell, A. (eds.). Contributions to mathematical economics in honor of Gerard Debreu. New York: North Holland 1986

Koopmans, T. C.: Objectives, constraints and outcomes in optimal growth models. Econometrica **35**, 1–15 (1967)

Kurz, M.: Optimal economic growth and wealth effects. Int. Econ. Rev. **9**, 348–357 (1968)

Li, T., Yorke, J.: Period three implies chaos. Am. Math. Monthly **82**, 985–992 (1975)

Majumdar, M., Nermuth, M.: Dynamic optimization in non-convex models with irreversible investment: monotonicity and turnpike results. Z. Nationalökonomie **42**, 339–362 (1982)

McKenzie, L.: Accumulation programs of maximum utility and the von Neumann facet. In: Wolfe, J. N. (ed.). Value, capital and growth, Edinburgh:University Press, pp. 353–383, 1968

McKenzie, L.: A primal route to the turnpike and Liapounov stability. J. Econ. Theory **27**, 194–209 (1982)

McKenzie, L. W.: Turnpike theory, discounted utility, and the von Neumann facet. J. Econ. Theory **30**, 330–352 (1983)

McKenzie, L. W.: Optimal economic growth: turnpike theorems and comparative dynamics. In: Arrow, K. J., Intriligator, M. D. (eds.). Handbook of mathematical economics, vol. III. Amsterdam: North-Holland 1986

Mitra, T., Ray, D.: Dynamic optimization on a non-convex feasible set: some general results for non-smooth technologies. Z. Nationalökonomie **44**, 151–175 (1984)

Nishimura, K., Yano, M.: Optimal chaos, nonlinearity and feasibility conditions. Econ. Theory **4**, 689–704 (1994)

Rasband, S. N.: Chaotic dynamics of nonlinear systems. New York: John Wiley 1990

Ross, S. M.: Introduction to stochastic dynamic programming. New York: Academic Press, Inc. 1983

Samuelson, P. A.: Optimality of profit-including prices under ideal planning. Proc. Nat. Acad. Sci. USA **70**, 2109–2111 (1973)

Wan, H. Y.: Optimal evolution of tree-age distribution for a tree farm. In: Castillo-Chavez, C. et. al. (eds.). Mathematical approaches to ecological and environmental problems. Lect. Notes Bio-math. Berlin: Springer 1988